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Two-dimensional skyrmions on the sphere

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Abstract: We study a model for two-dimensional skyrmions on a sphere of radius L. Such model simulates a skyrmion lattice of density $W/(2\pi L^2)$, where W is the skyrmion winding number. We show that, to a very good approximation, physical results depend only on the product αL^4, where α is the strength of potential term. In the range $\alpha L^4 \lesssim 3$ the order parameter vanishes, there is a uniform distribution of the density over the whole surface and the energy of the $W = 2$ sector lies above twice the energy of the $W = 1$ sector. If $\alpha L^4 \gtrsim 6$ the order parameter approaches unity and the density concentrates near one of the poles. Moreover the disoliton is always bound. We also present a variational solution to the field equations for which the pure αL^4-dependence is exact. Finally, some consequences of our results for the Quantum Hall Effect are discussed.

Keywords: Field Theories in Lower Dimensions, Sigma Models, Solitons Monopoles and Instantons.

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Although skyrmions were originally introduced to describe baryons in the context of non-linear meson theories [1], they are playing an increasing role in other areas of physics as well. In the present paper we consider the case of "baby skyrmions", i.e. skyrmions in two spatial dimensions. For example, 2D skyrmions are used to model the bubbles that appear in certain condense matter systems in the presence of an external magnetic field ; they could provide a mechanism associated with the disappearance of antiferromagnetism and the onset of HT_c superconductivity; etc.. It has been suggested [2] that they may also be responsible for many features of the Quantum Hall Effect (QHE). In addition, baby skyrmions have been also studied [3] in the context of strong interactions as a toy-model in order to understand the more complicated dynamics of usual skyrmions which live, of course, in 3 spatial dimensions.

For many of these applications it is of great interest to consider not only the properties of few baby skyrmion systems but also the behavior of such properties as the density of skyrmions increases. This problem has been studied by putting baby skyrmions on a lattice, which is, of course, a rather cumbersome numerical task (see, e.g., ref. [4]). Analogous studies have been done for 3D skyrmions to understand baryon properties in dense matter and the appearance of phase transitions such as chiral restoration, deconfinement, etc. [5]. However, in the 3D case it was realized that many of the qualitative results of the lattice calculations (and even some quantitative ones) could be obtained in a much easier way by studying the behavior of few skyrmions on a hypersphere [6]. This approach was introduced in ref. [7] and proposes to replace the periodic array in flat space $\mathbf{R}^3$ by a one (or few) skyrmion system on the compact manifold $\mathbf{S}^3(L)$ of radius L. The finite baryon number on $\mathbf{S}^3(L)$ corresponds to a finite baryon density, so that one obtains a model for skyrmion matter which is much simpler to study than any lattice model. Very recently [8], these ideas have been applied to a particularly simple 2D model in which the quadratic term in the derivatives is absent. In QHE this would correspond, after an appropiate scaling, to the limit in which the magnetic field goes to infinity. In many practical applications, however, it is not at all clear whether such limit can be taken. Therefore, in the present paper we will extend previous studies to a more general model for baby skyrmions on the sphere.

Baby skyrmions are obtained as non-trivial solutions of the well-known non-linear O(3) model. This model consists of three real scalar fields ϕ_a $(a = 1, 2, 3)$ subject to the constraint $\vec{\phi} \cdot \vec{\phi} = 1$. The equations of motion admit static solutions with finite energy which represent a mapping of $\mathbf{R}^2_{spat}$ into S^2_{int}. They are characterized by the winding number W and the density ρ

$$W = \frac{1}{8\pi} \int d^2r\, \rho; \qquad \rho \equiv \epsilon_{ij}\ \vec{\phi} \cdot \left(\partial^i \vec{\phi} \times \partial^j \vec{\phi} \right), \tag{1}$$

where i, j stand for the spatial coordinates. We start from the energy functional

$$E = E^{(2)} + E^{(4)} + E^{(p)}$$
$$E^{(2)} = \frac{1}{2} \int d^2r\ \partial_i \vec{\phi} \cdot \partial^i \vec{\phi}\,;\ E^{(4)} = \frac{1}{8} \int d^2r\ \rho\ \rho\,;\ E^{(p)} = \frac{\alpha}{2} \int d^2r\ (\hat{n}_3 - \vec{\phi})^2\,, \tag{2}$$

where $\hat{n}_3$ is a unit vector in the third direction in internal space and α is a parameter that is assumed positive. This type of energy functional is obtained as the static limit of either the relativistic $O(3)$ model or the non-relativistic Chern-Simons theory.

In 2D the quadratic term in the field derivatives gives a scale independent contribution to the soliton mass, due to the well known compensation between the derivatives and the integration [9]. Thus, the stabilization procedure for baby skyrmions is usually based on a competition between a quartic term and a linear potential term. This last (attractive) interaction tends to concentrate the skyrmion at $\phi_3 = 1$, yields a contribution to the total energy proportional to the square of the soliton size λ and destroys the original O(3) symmetry. In condense matter physics this potential is usually related to the magnetic interactions (e.g. Zeeman term in QHE). In the framework of hadron physics it would correspond to the explicit chiral symmetry breaking driven by the pion mass term and therefore considered to be only a small perturbation, and often neglected. The collapse of the field is prevented by means of the quartic term which yields a contribution proportional to $1/\lambda^2$. In QHE this contact interaction would be a "poor man" mocking of the Coulomb repulsion, while in hadron physics it can be understood as a large mass limit of the ρ-meson exchange. It should be mentioned that, as in the 3D case, the quartic term can be replaced by higher order derivative terms.[1] From the experience in 3D we do not expect major changes in the behavior of most of the quantities to be discussed below.

The simple relation between the coefficients of the quadratic and quartic terms in (2) is obtained through a convenient choice of the unit of length. Therefore, in flat space there is only one parameter in the model, namely the strength α of the potential term, aside from an overall multiplying factor which is already omitted in (2).

It is interesting to note that using the inequality

$$0 \leq \int d^2 r \left[\frac{1}{4}\left(\partial_i \vec{\phi} \pm \epsilon_{ij} \vec{\phi} \times \partial^j \vec{\phi}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\epsilon_{ij}\partial^i \vec{\phi} \times \partial^j \vec{\phi} \pm \sqrt{\alpha}(\hat{n}_3 - \vec{\phi})\right)^2 \right] \qquad (3)$$

one can find the Bogomol'nyi bound for the baby skyrmion energy. One obtains

$$E \geq 4\pi k \, \left(1 + \sqrt{\alpha}\right), \qquad (4)$$

which improves over the usual bound [3] through the contribution proportional to $\sqrt{\alpha}$.

As mentioned above, our aim is to extend the model above by going from $\mathbf{R}^2_{sp}$ to $\mathbf{S}^2_{sp}(L)$ where L is the radius of the two-sphere. A convenient choice of coordinates on $\mathbf{S}^2_{sp}(L)$ is given by the conventional polar coordinates θ, φ ($0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$)

$$x = L \sin\theta \cos\varphi \, ; \quad y = L \sin\theta \sin\varphi \, . \qquad (5)$$

[1]In fact, using particular combinations of derivative terms it is possible even to stabilize the soliton in the absence of the potential term. Although interesting, we do not consider this possibility here. For details, see ref.[10].

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The jacobian and the metric associated with the polar coordinates are:

$$\begin{aligned} J &= -L^2 \sin\theta \,, \\ ds^2 &= L^2(\sin^2\theta d\varphi^2 + d\theta^2)\,. \end{aligned} \tag{6}$$

In order to obtain explicit static solutions in the winding number $W = k$ sector we introduce the hedgehog parameterization

$$\phi_1 = \sin f\, \cos k\varphi\,; \quad \phi_2 = \sin f\, \sin k\varphi\,; \quad \phi_3 = \cos f\,, \tag{7}$$

where the "radial" profile $f = f(\theta)$ is subject to the boundary conditions $f(0) = \pi$ and $f(\pi) = 0$.

Using (7) and (6) we may express E_k, the soliton energy divided by $4\pi k$, in terms of the profile f as

$$E_k = \frac{1}{4k}\int d\theta\, \sin\theta \left[f'^2 + k^2\left(\frac{\sin f}{\sin\theta}\right)^2 + \frac{k^2}{L^2} f'^2 \left(\frac{\sin f}{\sin\theta}\right)^2 + 2\alpha L^2(1-\cos f)\right], \tag{8}$$

while the winding number density results

$$\rho_k = -\frac{2k}{L^2}\, f'\, \frac{\sin f}{\sin\theta}\,. \tag{9}$$

In eqs. (8), (9) a prime denotes differentiation with respect to θ.

Minimizing E_k we obtain the eq. of motion for f. It results

$$\begin{aligned} &\left[1 + \frac{k^2}{L^2}\left(\frac{\sin f}{\sin\theta}\right)^2\right] f'' + \\ &+ \left[f' - k^2\frac{\sin 2f}{\sin 2\theta} + \frac{k^2}{L^2} f' \frac{\sin f}{\sin\theta}\left(f'\frac{\cos f}{\cos\theta} - \frac{\sin f}{\sin\theta}\right)\right]\cot\theta - \alpha L^2 \sin f = 0\,. \end{aligned} \tag{10}$$

Figure 1(a) displays the profile function $f(\theta)$ in the $k = 1$ sector for different values of L, and $\alpha = 0.1$. Figure 1(b) shows the same magnitudes but for different values of α, and $L = 2$. For very dense systems or for very small values of α the identity map $f = \pi - \theta$ constitutes a valid (although not exact) solution for f. As shown in figs. 1(c) and (d), it corresponds to a uniform density ρ_1 over the surface of the sphere. As L or α increases, both the profile and the density become more concentrated near $\theta \approx 0$. The fact that similar results are obtained by varying either L or α raises the question whether the baby skyrmion on the sphere is a single-parameter model. The origin of such uniqueness may be traced back to the fairly good independence of the quadratic term $E^{(2)}$ respective to both L and α . This can be clearly seen in figs. 2(a) and (b). There it is shown the behaviour of the different contributions to the soliton energy as function of L and α, respectively. As a consequence, the physics of the model is determined only by the interplay between the quartic and the potential term, and thus depends basically on the product αL^4. The fact that some quantities (e.g. the mean

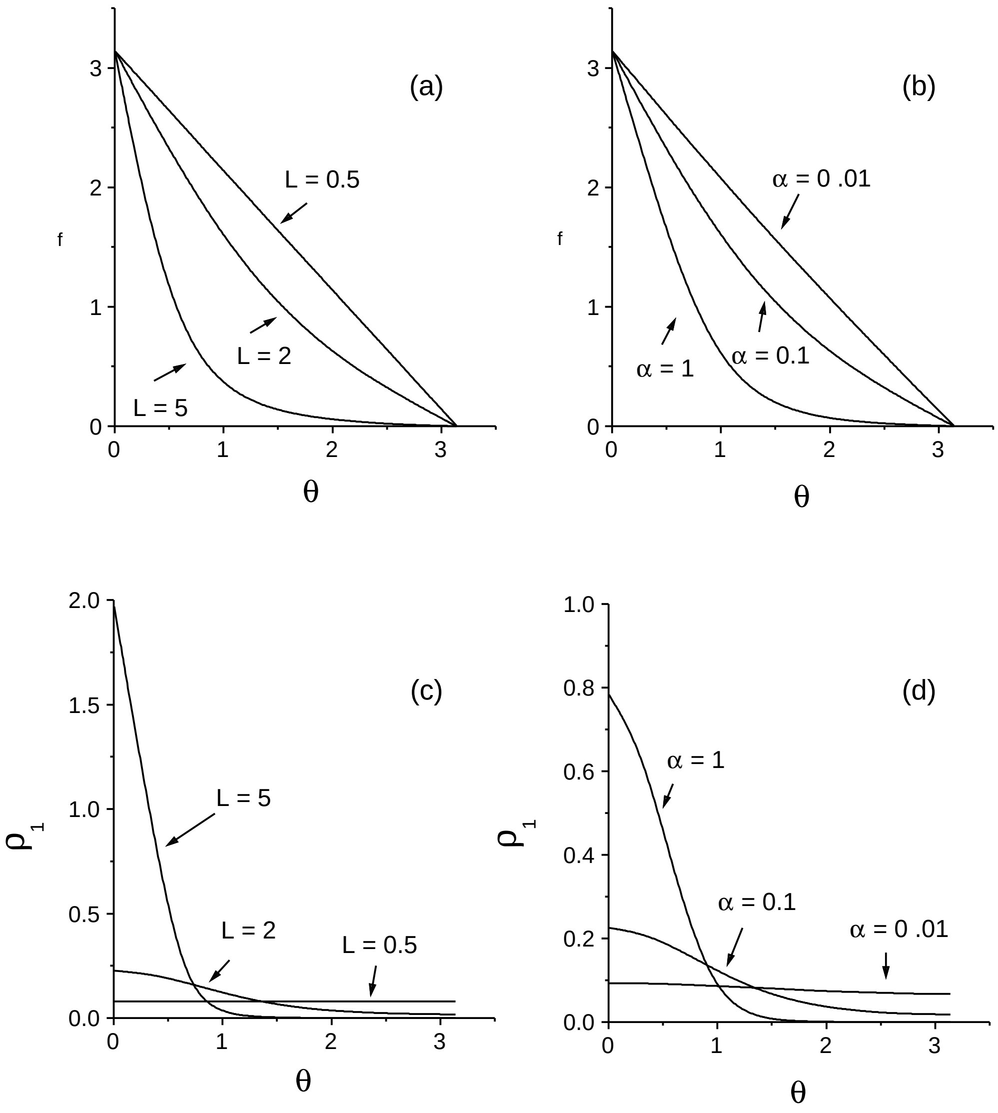

Figure 1: Soliton profiles (a) and (b) and winding number densities (c) and (d). The densities (9) are plotted in units of $8\pi/L^2$.

square radius in fig. 2) behave differently as function of L or as function of α is simply due to explicit dependencies on L. It is clear from fig. 2 that we may distinguish two regions which display different behaviors of the physical magnitudes. For large values of αL^4 the order parameter defined as the mean value of ϕ_3

$$\bar{\phi}_3 = \frac{1}{2}\int d\theta \sin\theta \;\phi_3 \tag{11}$$

tends to 1; the quartic and potential contributions $E^{(4)} < E^{(p)}$ become rather similar; the energy of the $k = 2$ sector is smaller than (twice) the one of the $k = 1$ sector; the mean square radius

$$\bar{r}_k = \left[2\pi L^2 \int d\theta \sin\theta \;\rho_k \;(\sin\theta)^2\right]^{1/2} \tag{12}$$

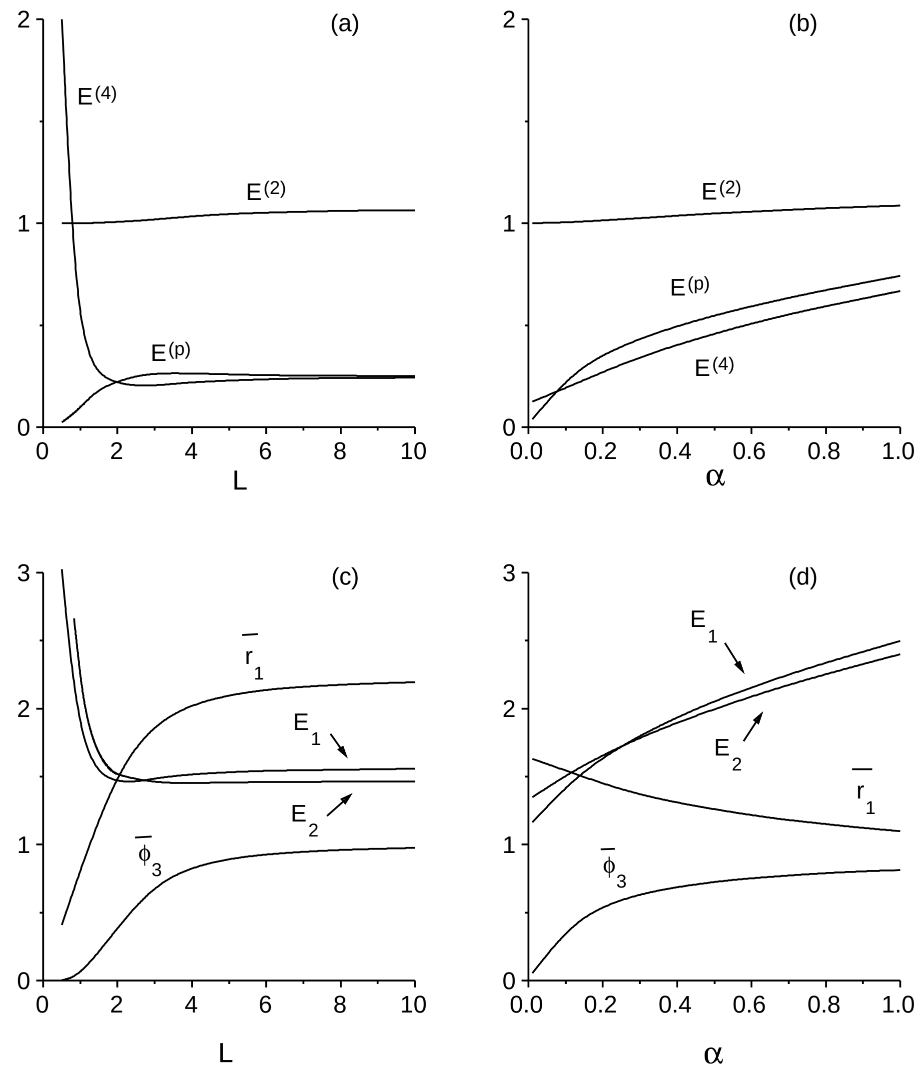

Figure 2: Various properties of the solution. (a) and (b) contain the different contributions (2) to the soliton energy for winding number $k = 1$ (in units of 4π). (c) and (d) include the total soliton energies (8), the order parameter (11) and the mean square radius (12).

decreases, if plotted in units of L. On the contrary, for small values of αL^4 the order parameter vanishes; $E^{(4)} \gg E^{(p)}$; the energy of the $k = 2$ sector is larger than (twice) the one of the $k = 1$ sector; the mean square radius remains approximately constant. Between the two extreme behaviors there is a broad transition region corresponding approximately to the values $3 \lesssim \alpha L^4 \lesssim 6$. This differs from the situation in the absence of the quadratic term $E^{(2)}$[8].

The behaviour of the physical quantities as function of L is, in general, similar to the one obtained in 3D [6]. Due to the presence of the potential term, however, the sharp transition found in 3D (in the chiral limit) turns out to be a broad one here. Such term is also responsible for the absence of "swelling" of the mean square radius in our model. This seems to be the only qualitative feature of 3D skyrmions which is not shared by baby skyrmions.

In the context of QHE there is an undergoing discussion about the stability of the disoliton at low densities, with respect to the decay into two $k = 1$ solitons. In ref. [11], it was found that the disoliton is stable independently of the strength of the potential term. This seems to be in contradiction with ref. [12], where it is shown that there is a critical strength above which the disolitons become unstable. Our results indicate that at zero density the disoliton is always stable. They also confirm the conjeture made in ref. [11] that, as density increases (and therefore L decreases), there might be a transition above which disolitons become unstable. The fact that at zero density (flat space) the disoliton is stable is in agreement with the result obtained in 3D. There the disoliton is supposed to represent the deuteron. As well-known, at classical level, the lowest diskyrmion (the doughnut) is rather strongly bound [13]. Recently [14] it has been shown that quantum effects tend to decrease such binding. It is clear that in order to make more solid statements such quantal effects should be introduced in our model through a proper quantization method (see, for instance, ref. [15]). Another point to take into account when applying our model to QHE is the long range nature the Coulomb interaction. As mentioned above a very crude approximation has been used in the present work. Namely, we use a local density-density interaction as a four-derivative term. The consequences of replacing such term by a finite range interaction is currently under investigation.

We finish by presenting an approximate variational solution for the baby skyrmions on the sphere. For this purpose one can use the solutions of the equation of motion (10) in the absence of quartic and potential terms. They can be written as

$$\tan\frac{f}{2} = 2\left[\frac{\eta}{2}\cot\frac{\theta}{2}\right]^k , \tag{13}$$

where η is an arbitrary constant. These solutions reduce to the known expression in flat space [9] and saturate the corresponding Bolomol'nyi bound (i.e. Eq.(4) with $\alpha = 0$) for any value of η. Therefore when used as variational solution of the full model the value of η turns out be only a function of the combination αL^4. For example, minimization of the soliton energy in the $k = 1$ sector leads to the following equation for η:

$$\alpha\, L^4 = \frac{(\eta^2-1)^4(\eta^2+1)}{24\eta^4[\eta^2-1-(\eta^2+1)\log\eta]} . \tag{14}$$

All the features that are present in the exact solution of the hedgehog ansatz are qualitatively reproduced, the quantitative agreement lying within few percents.

To conclude, we have studied the consequences of locating one or two baby skyrmions on the surface of a sphere of radius L. Thus the model covers the whole range between the two limits of low densities (flat space) and of high density (small L). The soliton is stabilized through the presence of a (repulsive) quartic term in the field derivatives and of an (attractive) potential term proportional to the strength α of an external field, in addition to the usual quadratic term. A value of the Bogomol'nyi bound which is higher than the one currently used in the literature on baby skyrmions has been obtained. Within the hedgehog approximation, the model was solved exactly. Although in principle the model has two independent parameters, we show that the product αL^4 determines the physically meaningful properties of the system. The solution displays two characteristic regimes: the order parameter vanishes and there is a uniform distribution of the density over the whole surface for the range $\alpha L^4 \lesssim 3$; on the contrary, the order parameter approaches unity and the density concentrates near $\theta \approx 0$ if $\alpha L^4 \gtrsim 6$. In the first region the energy of the $k = 2$ sector lies above twice the energy of the $k = 1$ sector. The opposite is true for the soliton region, in which the disoliton is *always* bound. Finally, a variational solution that realizes exactly the pure αL^4-dependence is given. We believe that some of the results obtained in the present work can be directly extended to interesting issues such as phase transitions in Quantum Hall Effect. There, the skyrmion density would be related to the departure of the filling factor from one. However, further investigations on e.g. the influence of the range of the effective interaction have to be made before drawing stronger conclusions. We hope to be able to report on these matters in forthcoming publications.

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